

# Superradiance Problem of Bosons and Fermions for Rotating Black Holes in Bargmann-Wigner Formulation

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## Abstract

Bargmann-Wigner equations are formulated to represent bosonic fields in terms of fermionic fields in curved spacetime. The superradiance phenomena of bosons and fermions in rotating black hole spacetime are studied in the Bargmann-Wigner formulation. As a result of the consistent description between scalar bosons and spinor fermions, superradiance phenomena of the type of positive frequency ( $0 < \omega$ ) and negative momentum near horizon ( $p_H < 0$ ) are shown not to occur.

## 1 Introduction

Recently many black holes have been observed and most of all are considered as rotating black holes. Many theoretical analysis have been done focused on matter behavior around rotating black hole spacetime. The framework of the general relativistic theory is a reliable approach to the matter perturbation outside the black hole horizon in four dimensions.

As one of standard radiation problems of matter fields around rotating black holes, the outstanding super-radiance problem may occur in which the reflected intensity becomes stronger than the incident intensity. The successive occurrences of superradiance phenomena will cause the serious problem of the instability of black holes [1, 2, 3, 4].

Under Kerr background spacetime, Klein-Gordon, Dirac, Maxwell, Rarita-Schwinger and Einstein equations for massless fields are known to reduce to the separable one component field equations, which are called as the Teukolsky equations [5]. The analytic perturbative solutions of the Teukolsky equations show that the reflected intensity can become stronger than the incident intensity for bosons and the super-radiance occurs for strongly rotating black hole spacetime. The situation is summarized according to the work by S. Mano and E. Takasugi [6] as follows:

$$|A_s^{(\text{inc})}|^2 = \frac{2\omega^{2s}}{|C_s|^2} |A_s^{(\text{ref})}|^2 + \delta_s |A_s^{(\text{trans})}|^2 ,$$

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where  $A_s^{(\text{inc})}$ ,  $A_s^{(\text{ref})}$  and  $A_s^{(\text{trans})}$  denote the amplitudes of incident, reflected and transmitted waves respectively with spin  $s (= 0, 1/2, \dots, 2)$  and frequency  $\omega$ . The Starobinsky constants are denoted by  $C_s$  and coefficients  $\delta_s$  are defined:

$$\delta_s = -i \exp(i\pi s) \left( \frac{\omega}{\sqrt{M^2 - a^2}} \right)^{2s-1} \frac{\Gamma(1 - s + 2i\epsilon_+)}{\Gamma(s + 2i\epsilon_+)},$$

where  $M, a$  denote the mass and angular momentum of Kerr black holes. The constant  $\epsilon_+$  is defined as

$$\epsilon_+ = (1 - (a/M))^{-1/2} r_H p_H,$$

where  $r_H$  and  $p_H$  denote the radius of the event horizon of black holes and momentum of fields near horizon, which is defined by

$$r_H = M + \sqrt{M^2 - a^2}, \quad p_H = \omega - m\Omega_H,$$

with the angular velocity of black holes  $\Omega_H = a/(r_H^2 + a^2)$ . The coefficients  $\delta_s$  determine whether the super-radiance occurs or not according to their sign. Those for fermions  $\delta_{1/2}, \delta_{3/2}$  are shown to be positive definite values while those for bosons

$$\delta_0, \delta_1, \delta_2 \propto p_H,$$

can be negative for large angular momentum of black holes. That is, the super-radiance occurs for bosons in case of  $p_H < 0$ , but not for fermions.

We list up some special features in the super-radiance phenomena of rotation black holes as follows:

- 1) The super-radiance may occur for bosonic fields to the strongly rotating black holes (or light mass black holes) but not to the weakly rotating black holes (or heavy mass black holes) in four dimensional spacetime [1, 7, 8, 9].
- 2) The super-radiance do not occur for fermionic fields to any rotating black holes in four dimensional spacetime for massive fields [10] as well as massless fields [11].
- 3) In the three dimensional case, the super-radiance phenomena have been shown not to occur [12, 13].

The purpose of this paper is to resolve the puzzle in super-radiance phenomena for fermions and bosons using the Bargmann-Wigner formulation, which connect fermions and bosons through the Bargmann-Wigner equations. As a result of our study, we obtain the result that the momentum near horizon cannot be negative, therefore the superradiance for scalar bosons  $0 < \omega$  and  $p_H = \omega - m\Omega_H < 0$  does not occur, which is in coincident with those for fermions.

The organization of this paper is as follows. In section 2, the Bargmann-Wigner equations in flat spacetime are reviewed briefly in the sake of the following study. In section 3, the Bargmann-Wigner equations are extended and formulated in general curved spacetime in the case of bi-spinor fields. In section 4, the Bargmann-Wigner formulation is applied to the fermion and boson puzzle in super-radiance phenomena in Kerr geometry. Summary is given in the final section.

## 2 Bargmann-Wigner formulation in flat spacetime

In this section, we review briefly the original Bargmann-Wigner equations in flat spacetime. An explicit realization is the case for spin 1 states [14, 15].

### 2.1 Original Bargmann-Wigner equations in flat spacetime

We start to short review of the original Bargmann-Wigner equations, which are aimed to get a general system of relativistic wave equations for higher spin states

$$\begin{aligned} (\gamma^\lambda \partial_\lambda + \mu)_{\alpha\alpha'} \Psi(x)_{\alpha'\beta\dots}^{(\text{BW})} &= 0 \\ (\gamma^\lambda \partial_\lambda + \mu)_{\beta\beta'} \Psi(x)_{\alpha\beta'\dots}^{(\text{BW})} &= 0 \\ &\dots, \end{aligned} \quad (2.1)$$

where  $\mu$  denotes the mass of particles and multi-spinor fields  $\Psi^{(\text{BW})}$  are assumed to be symmetric with respect to their spinor suffices:  $\Psi(x)_{\alpha\beta\dots}^{(\text{BW})} = \Psi(x)_{\beta\alpha\dots}^{(\text{BW})}$ .

### 2.2 Bargmann-Wigner equations for spin 1 states

For spin 1 states, the Bargmann-Wigner equations reduce to two equations as follows

$$(\gamma^\lambda \partial_\lambda + \mu) \Psi(x)^{(\text{BW})} = 0 \quad (2.2)$$

$$\Psi(x)^{(\text{BW})} (\overleftarrow{\partial}_\lambda \gamma^{\lambda T} + \mu) = 0, \quad (2.3)$$

where  $T$  denotes the transpose operation. We write the bi-spinor field in the boson expansion form:

$$\Psi(x)^{(\text{BW})} = \sqrt{\mu} \gamma^\lambda C A_\lambda(x) + \frac{1}{2\sqrt{\mu}} \Sigma^{\lambda\tau} C F_{\lambda\tau}(x), \quad (2.4)$$

where  $A_\lambda$  and  $F_{\lambda\tau}$  are vector and anti-symmetric second-rank tensor fields and  $\Sigma^{\lambda\tau}$  and  $C$  are the spin tensor and the charge conjugation matrix respectively. We apply the Bargmann-Wigner equations and get the coupled equations for bosons:

$$F_{\lambda\tau}(x) = \partial_\lambda A_\tau(x) - \partial_\tau A_\lambda(x), \quad (2.5)$$

$$\partial^\lambda F_{\lambda\tau} = \mu^2 A_\tau(x), \quad (2.6)$$

which derive the massive vector field equations. The Bargmann-Wigner system shows the consistency between fermionic expression in eqs.(2.2)-(2.3) and bosonic expression in eqs.(2.5)-(2.6), which represent the same objects from two sides.

In the following study, it is convenient to introduce to define the modified second rank Bargmann-Wigner field:

$$\Psi(x) = \Psi(x)^{(\text{BW})} C^{-1} \gamma_5 \quad (2.7)$$

to avoid the transpose and charge conjugation operations as

$$(\gamma^\lambda \partial_\lambda + \mu) \Psi(x) = 0, \quad (2.8)$$

$$\Psi(x) (\overleftarrow{\partial}_\lambda \gamma^\lambda + \mu) = 0. \quad (2.9)$$

### 3 Bargmann-Wigner formulation in curved spacetime

Next we develop the Bargmann-Wigner system, establish the Lagrangian formalism and derive the current conservation law in curved spacetime.

#### 3.1 Notation and definition in local Minkowski and curved spacetime

In curved spacetime, the gamma matrices and the algebra are defined in local Minkowski space with the Latin letters for suffix notation [16, 17]<sup>1</sup>:

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij}, \quad \eta^{ij} = \text{diag}(-1, 1, 1, 1). \quad (3.1)$$

The gamma matrices and the algebra in curved spacetime are obtained from those in local Minkowski spacetime with the Greek letters for suffix notation:

$$\gamma^\mu := b_i^\mu \gamma^i, \quad (3.2)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g^{\mu\nu} = b^{i\mu} b_i^\nu, \quad (3.3)$$

where  $b_i^\mu$  denote the vierbein, which connect the physical quantities in flat Minkowski spacetime with those in curved spacetime, and  $g^{\mu\nu}$  denote the metric tensor in curved spacetime.

The spinor fields  $\psi(x)$  are introduced to transform as spinors under local Lorentz transformations in local Minkowski spacetime and scalars under general coordinate transformations in curved spacetime:

$$\psi(x) \rightarrow D(\Lambda(x))\psi(x), \quad (3.4)$$

where  $D(\Lambda)$  is the spinor representation of the homogeneous Lorentz group. A covariant derivative for spinors is introduced as

$$\mathcal{D}_\mu \psi = (\partial_\mu + \Omega_\mu)\psi, \quad (3.5)$$

which is defined to transform under local Lorentz transformations like  $\psi$  itself:

$$\mathcal{D}_\mu \psi(x) \rightarrow D(\Lambda(x))\mathcal{D}_\mu \psi(x). \quad (3.6)$$

The connection matrix can be written in the form

$$\Omega_\mu = \frac{1}{2}\omega_{,\mu}^{ij}\Sigma_{ij}, \quad (3.7)$$

where  $\omega_{,\mu}^{ij}$  is the spin connection and  $\Sigma_{ij}$  are the spin matrices representing the generators of homogeneous Lorentz group and for spinors they are written as

$$\Sigma_{ij} := \frac{1}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i). \quad (3.8)$$

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<sup>1</sup>See appendix for the explicit expression of gamma matrices.

The covariant Dirac equation in curved spacetime is derived using the defined notations as

$$(\gamma^\mu(\partial_\mu + \Omega_\mu) + \mu)\psi(x) = 0 , \quad (3.9)$$

where  $\mu$  is the mass parameter.

For vector fields  $A_\nu(x)$  transforming like vectors under general coordinate transformations, a covariant derivative is defined as

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\nu\mu}^\lambda A_\lambda , \quad (3.10)$$

where  $\Gamma_{\nu\mu}^\lambda$  denote the affine connection.

The covariant derivatives for vierbeins with suffix of general curved and local Minkowski spacetime are defined:

$$\mathcal{D}_\mu b^i_\nu = \partial_\mu b^i_\nu + \omega^i_{j,\mu} b^j_\nu - \Gamma_{\mu\nu}^\lambda b^i_\lambda . \quad (3.11)$$

In order to determine the geometry the vierbein condition is imposed:

$$\mathcal{D}_\mu b^i_\nu = 0 , \quad (3.12)$$

which leads the explicit form of the spin connection

$$\omega^{ij}_{,\mu} = g^{\nu\lambda} b^i_\nu (\partial_\mu b^j_\lambda - \Gamma_{\lambda\mu}^\tau b^j_\tau) . \quad (3.13)$$

### 3.2 Bargmann-Wigner equations for spin 0 and 1 states in curved spacetime

We consider the bi-spinor Bargmann-Wigner fields  $\Psi(x)$  as spin 0 and 1 bosonic states taking off the symmetry restriction with respect to spinor suffices. The bi-spinor fields transform under local Lorentz transformations like spinors from lefthand side and righthand side as

$$\Psi(x) \rightarrow D(\Lambda(x))\Psi(x)D^{-1}(\Lambda(x)) , \quad (3.14)$$

which define the covariant derivatives for bi-spinors as

$$\mathcal{D}_\mu \Psi(x) := \partial_\mu \Psi + \Omega_\mu \Psi - \Psi \Omega_\mu . \quad (3.15)$$

Then the covariant field equations for bi-spinors are derived

$$(\gamma^\mu \mathcal{D}_\mu + \mu)\Psi(x) = 0 , \quad (3.16)$$

$$\Psi(x)(\overleftarrow{\mathcal{D}}_\mu \gamma^\mu + \mu) = 0 . \quad (3.17)$$

We expand the bi-spinor field in a set of bosons as

$$\Psi(x) = \sqrt{\mu}(S(x) + \gamma_5 P(x) - \gamma^\mu V_\mu(x) + \gamma_5 \gamma^\mu A_\mu(x)) + \frac{1}{2\sqrt{\mu}} \gamma_5 \Sigma^{\mu\nu} F_{\mu\nu}(x) , \quad (3.18)$$

where  $S, P, V_\mu, A_\mu$  and  $F_{\mu\nu}$  denote the scalar, pseudoscalar, vector and tensor fields respectively. We apply the Bargmann-Wigner equations in curved spacetime (3.16) and (3.17). Adding these equations, we find the set of relations among bosons

$$\mu S(x) = \nabla^\lambda V_\lambda(x) \quad (3.19)$$

$$\mu V_\lambda(x) = \partial_\lambda S(x) \quad (3.20)$$

$$P(x) = 0 \quad (3.21)$$

$$\mu^2 A_\lambda(x) = \nabla^\tau F_{\tau\lambda}(x) \quad (3.22)$$

$$F_{\lambda\tau}(x) = \nabla_\lambda A_\tau(x) - \nabla_\tau A_\lambda(x). \quad (3.23)$$

For independent spin 0 and 1 fields, the Klein-Gordon type field equations in curved spacetime are obtained as in the flat spacetime:

$$(\nabla^\mu \partial_\mu - \mu^2)S(x) = 0, \quad (3.24)$$

$$\nabla^\mu (\nabla_\mu A_\nu - \nabla_\nu A_\mu) - \mu^2 A_\nu(x) = 0. \quad (3.25)$$

The difference of the Bargmann-Wigner equations (3.16)-(3.17) is calculated to be

$$\begin{aligned} & (\gamma^\nu \mathcal{D}_\nu + \mu)\Psi - \Psi(\overleftarrow{\mathcal{D}}_\nu \gamma^\nu + \mu) \\ &= 2\sqrt{\mu}(\gamma_5 \gamma^\nu \partial_\nu P + \Sigma^{\nu\lambda} \nabla_\nu V_\lambda - \gamma_5 \nabla^\nu A_\nu) - \frac{1}{\sqrt{\mu}} \gamma_5 \Sigma^{\nu\lambda\tau} \nabla_\nu F_{\lambda\tau}, \end{aligned} \quad (3.26)$$

where  $\Sigma^{\nu\lambda\tau}$  denotes the third rank completely anti-symmetric tensor of gamma matrices. For each term in eq.(3.26), we use bosonic field relations in eqs.(3.19)-(3.23) and obtain

$$\Sigma^{\nu\lambda} \nabla_\nu V_\lambda = \frac{1}{2\mu} \Sigma^{\nu\lambda} (\Gamma_{\nu\lambda}^\tau - \Gamma_{\lambda\nu}^\tau) \partial_\tau S, \quad (3.27)$$

$$\nabla^\nu A_\nu = \frac{1}{2\mu^2} g^{\nu\rho} g^{\lambda\sigma} (R_{\rho\lambda\nu}^\tau F_{\tau\sigma} + R_{\sigma\lambda\nu}^\tau F_{\tau\rho} + (\Gamma_{\nu\lambda}^\tau - \Gamma_{\lambda\nu}^\tau) \nabla_\tau F_{\rho\sigma}), \quad (3.28)$$

$$\Sigma^{\nu\lambda\tau} \nabla_\nu F_{\lambda\tau} = \Sigma^{\nu\lambda\tau} (R_{\tau\nu\lambda}^\rho A_\rho + (\Gamma_{\nu\lambda}^\rho - \Gamma_{\lambda\nu}^\rho) \nabla_\rho A_\tau), \quad (3.29)$$

where  $R_{\rho\lambda\nu}^\tau$  denotes the Riemann-Christoffel curvature tensor. In order to satisfy the difference of the Bargmann-Wigner equations, we require the following conditions:

- (C1) The torsion of the background geometry should be zero:  $\Gamma_{\nu\lambda}^\rho - \Gamma_{\lambda\nu}^\rho = 0$ , for non-zero scalar field  $S(x)$ .
- (C2) Axial-vector field should vanish:  $A_\nu(x) = 0$  and therefore  $F_{\nu\lambda}(x) = 0$ , for general curved geometry of non-zero curvature.

We set the above two conditions (C1) and (C2) in the following of this paper <sup>2</sup>.

Here we stress that the solution relations between fermionic states and bosonic states. If the bosonic solutions  $S(x)$  and  $A_\mu(x)$  of equations (3.24) and (3.25) are obtained, they are automatically the bi-spinor field solutions  $\Psi(x)$  of (3.16) and (3.17). They are the same physical

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<sup>2</sup>For the bi-spinor field we impose the anti-symmetric spinor suffix:  $\Psi_{\alpha\beta}^{(BW)}(x) = -\Psi_{\beta\alpha}^{(BW)}(x)$  instead of the original Bargmann-Wigner field in which spinor suffix is completely symmetric.

objects from different sides. Furthermore the bi-spinor fields are transformed to a set of four Dirac spinors  $\psi^{(i)}$ , ( $i = 1 \sim 4$ ) as

$$\Psi(x)U(x) = (\psi^{(1)}(x), \psi^{(2)}(x), \psi^{(3)}(x), \psi^{(4)}(x)) , \quad (3.30)$$

where the local transformation  $U(x)$  is defined by  $U^{-1}\partial_\mu U = \Omega_\mu$ . Then each spinor field satisfies the Dirac equation:

$$(\gamma^\mu(\partial_\mu + \Omega_\mu) - \mu)\psi(x)^{(i)}(x) = 0 \quad (i = 1 \sim 4). \quad (3.31)$$

Therefore the existence of stable scalar bosonic solutions is directly related to the existence of stable spinor fermionic states. Both of bosonic and fermionic view points are the same from different sides and the inverse statement is also correct.

### 3.3 Lagrangian formalism and conserved current

In order to establish the Bargmann-Wigner formulation in curved spacetime, we introduce the Lagrangian formalism. The Lagrangian density for the bi-spinor fields is proposed to be

$$\mathcal{L} = -\frac{1}{8}\text{Tr}\{\bar{\Psi}(x)(\gamma^\mu \mathcal{D}_\mu + \mu)\Psi(x) + \Psi(x)(\overleftarrow{\mathcal{D}}_\mu \gamma^\mu + \mu)\bar{\Psi}(x)\} , \quad (3.32)$$

where the adjoint spinor is defined as

$$\bar{\Psi} = (-i\gamma_0)\Psi^\dagger(-i\gamma_0) . \quad (3.33)$$

From this Lagrangian density, the sum of the Bargmann-Wigner equations (3.16) and (3.17) are obtained. The Lagrangian density in the bi-spinor form reproduces the correct Lagrangian density in the bosonic form expressing the independent bosonic fields  $S(x)$  and  $A_\mu(x)$  as

$$\mathcal{L} = -\partial^\mu S^\dagger(x)\partial_\mu S(x) - \mu^2 S^\dagger(x)S(x) , \quad (3.34)$$

which leads the correct equation of motion.

From the Lagrangian, the conserved current is derived as the Noether current for the phase transformation to bi-spinor fields:

$$\Psi(x) \rightarrow \exp(i\alpha(x))\Psi(x) . \quad (3.35)$$

The invariance of the Lagrangian under the phase transformation leads the conserved current in the bi-spinor expression

$$J_\mu = -\frac{\delta \mathcal{L}}{\delta \partial_\mu \alpha(x)} = \frac{i}{4}\text{Tr} \bar{\Psi} \gamma_\mu \Psi , \quad (3.36)$$

and in the boson expression

$$J_\mu = -i(S^\dagger \partial_\mu S - \partial_\mu S^\dagger S) . \quad (3.37)$$

The conservation of the current can be shown using the field equations directly

$$\nabla^\mu J_\mu = 0 . \quad (3.38)$$

## 4 Application to super-radiance puzzle between bosons and fermions in rotating black holes

We apply the Bargmann-Wigner formulation, which was established in section 3, to the super-radiance puzzle between bosons and fermions. For this purpose, we first study a simple application to free spherical waves. Next we apply this result to the asymptotic infinity region of Kerr black hole geometry and derive the current relation between bosons and fermion near the event horizon to show no occurrence of the bosonic super-radiance.

### 4.1 Case of free spherical waves

To understand the relation between bosonic and fermionic description explicitly, we consider free spherical waves as a simple application in flat spacetime. For a scalar boson example, the solution of Klein-Gordon equation is expressed by a spherical wave in the polar coordinate as

$$S^{(0)}(x) = Y_\ell^m(\theta, \phi) R_\ell(r) \exp(-i\omega t), \quad (4.1)$$

where  $Y_\ell^m$ ,  $R_\ell$  and  $\omega$  denote the spherical harmonics, the radial wave function and the frequency respectively. The normalization factor for boson is assumed to be included in the radial function.

For this bosonic solution the corresponding bi-spinor solution is obtained through equations (3.18) and (3.20) in the matrix form as

$$\begin{aligned} \Psi^{(0)}(x) &= (\mu - \gamma^\lambda \partial_\lambda) \frac{1}{\sqrt{\mu}} S^{(0)}(x), \\ &= \frac{1}{\sqrt{\mu}} \begin{pmatrix} \mu + \omega & i\sigma \cdot \nabla \\ -i\sigma \cdot \nabla & \mu - \omega \end{pmatrix} Y_\ell^m(\theta, \phi) R_\ell(r) \exp(-i\omega t). \end{aligned} \quad (4.2)$$

We write this as a set of four spinors as

$$\Psi^{(0)}(x) = \sqrt{\frac{\omega + \mu}{\mu}} (\psi^{(1)}(x), \psi^{(2)}(x), \psi^{(3)}(x), \psi^{(4)}(x)), \quad (4.3)$$

where  $\psi^{(1)}$ ,  $\psi^{(2)}$  are spinors of the positive energy with spin up and down respectively and  $\psi^{(3)}$ ,  $\psi^{(4)}$  are those of the negative energy. The angular wave functions  $Y_\ell^m$  can be written by the combination of normalized spin-angular functions  $\mathcal{Y}$  as

$$Y_\ell^m(\theta, \phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{\frac{\ell + m + 1}{2\ell + 1}} \mathcal{Y}_{j, \ell}^{j_3}(\theta, \phi) - \sqrt{\frac{\ell - m}{2\ell + 1}} \mathcal{Y}_{j', \ell}^{j_3}(\theta, \phi), \quad (4.4)$$

with total angular momenta  $j = \ell + 1/2$ ,  $j' = \ell - 1/2$  and their azimuthal component  $j_3 = m + 1/2$ . Corresponding to the recombination, the positive energy and spin up spinor can be written as [19]

$$\psi^{(1)} = \sqrt{\frac{\ell + m + 1}{2\ell + 1}} \psi^{(+)}(x) - \sqrt{\frac{\ell - m}{2\ell + 1}} \psi^{(-)}(x), \quad (4.5)$$



where the spin parallel and antiparallel spinors  $\psi^{(\pm)}$  are denoted

$$\psi^{(+)} = \frac{1}{r} \begin{pmatrix} F(r) \mathcal{Y}_{j,\ell}^{j_3} \\ iG^{(+)}(r) \mathcal{Y}_{j,\ell+1}^{j_3} \end{pmatrix}, \quad \psi^{(-)} = \frac{1}{r} \begin{pmatrix} F(r) \mathcal{Y}_{j',\ell}^{j_3} \\ iG^{(-)}(r) \mathcal{Y}_{j',\ell-1}^{j_3} \end{pmatrix}. \quad (4.6)$$

The fermion radial functions are obtained by the explicit calculation of the operation  $-i\sigma \cdot \nabla$  in equation (4.2) as

$$\begin{aligned} F(r) &= \sqrt{\omega + \mu} r R_\ell(r), \\ G^{(+)}(r) &= \frac{(\partial_r - (\ell + 1)/r)F(r)}{\omega + \mu}, \quad G^{(-)}(r) = \frac{(\partial_r + \ell/r)F(r)}{\omega + \mu}, \end{aligned} \quad (4.7)$$

where the normalization factor for fermion relative to the boson  $\sqrt{\omega + \mu}$  is included in the radial function  $F(r)$ . Fermion normalization factor  $\sqrt{\omega + \mu}$  is included relative to bosonic one. Other spin states  $\psi^{(2 \sim 4)}$  can be obtained similarly.

We have established the relation between scalar wave function  $S$  and spinor  $\psi^{(1)}$  or  $\psi^{(\pm)}$  using the bi-spinor field  $\Psi$  explicitly in the spherical symmetric case. Next we consider the radial component of the conserved current given by

$$\begin{aligned} J_r^{(0)} &= \int \sqrt{-g} d\theta d\phi (-i)(S^{(0)*} \partial_r S^{(0)} - \partial_r S^{(0)*} S^{(0)}), \\ &= r^2 i(\partial_r R_\ell^* R_\ell - R_\ell^* \partial_r R_\ell), \end{aligned} \quad (4.8)$$

for scalar bosons and

$$\begin{aligned} J_r^{(k)} &= \int \sqrt{-g} d\theta d\phi i \bar{\psi}^{(k)} \gamma \cdot \frac{x}{r} \psi^{(k)} \\ &= r^2 i(\partial_r R_\ell^* R_\ell - R_\ell^* \partial_r R_\ell), \end{aligned} \quad (4.9)$$

for spinors with  $k = 1, +$  or  $-$ . Therefore we have obtained the current relation between scalar boson and spinor (spin up, spin parallel or spin antiparallel) as

$$J_r^{(0)} = J_r^{(1)} = J_r^{(+)} = J_r^{(-)}. \quad (4.10)$$

## 4.2 Super-radiance puzzle in Kerr black hole geometry

In order to solve the super-radiance puzzle applying the Bargmann-Wigner formulation, we study the scattering problems for fermionic and bosonic fields in Kerr black hole geometry, of which metric is represented in Boyer-Lindquist coordinates [18]:

$$ds^2 = \frac{\Delta}{\Sigma} [dt - a \sin^2 \theta d\phi]^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2) d\phi - a dt]^2, \quad (4.11)$$

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad (4.12)$$

where  $M, a$  denote the mass and angular momentum of the Kerr black hole respectively.

We first consider the scattering problem for the spin 0 scalar field in Kerr geometry. The scalar field can be written in the polar coordinate system as

$$S(x) = R(r)Y(\theta, \phi) \exp(-i\omega t), \quad (4.13)$$

where  $R(r)$  and  $Y(\theta, \phi)$  denote the radial and angular wave functions respectively, which obey the equations:

$$\left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta - (a\omega \sin \theta - \frac{m}{\sin \theta})^2 - \mu^2 a^2 \cos^2 \theta + \lambda \right) Y(\theta, \phi) = 0, \quad (4.14)$$

$$(\partial_r \Delta \partial_r + \frac{((r^2 + a^2)\omega - am)^2}{\Delta} - \mu^2 r^2 - \lambda) R(r) = 0, \quad (4.15)$$

where  $\mu$  and  $\lambda$  denote the mass of particle and separation parameter.

To study the behavior of radial wave function near infinity and event horizon, a new radial coordinate  $r^*$  is introduced:

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{r^2 - 2Mr + a^2}. \quad (4.16)$$

Using the new coordinate, radial field solutions in Kerr geometry become free waves near the spatial infinity  $r \rightarrow \infty$  ( $r^* \rightarrow \infty$ ) and event horizon  $r \rightarrow r_H = M + \sqrt{M^2 - a^2}$  ( $r^* \rightarrow -\infty$ ):

$$\sqrt{r^2 + a^2} R(r) \sim \begin{cases} \omega^{-1/2} (A_B^{(\text{inc})} \exp(-ip_\infty r^*) + A_B^{(\text{ref})} \exp(ip_\infty r^*)), & (r \rightarrow \infty) \\ |p_H|^{-1/2} A_B^{(\text{trans})} \exp(-ip_H r^*), & (r \rightarrow r_H) \end{cases}, \quad (4.17)$$

where  $A_B^{(\text{inc})}$ ,  $A_B^{(\text{ref})}$  and  $A_B^{(\text{trans})}$  denote scalar boson amplitudes of incident wave, reflected and transmitted waves respectively. Bosonic normalization factors are included in front of each amplitude. Momenta near the infinity and event horizon are denoted by

$$p_\infty = \sqrt{\omega^2 - \mu^2} \quad \text{and} \quad p_H = \omega - \Omega_H m \quad (4.18)$$

which particle mass  $\mu$ , azimuthal momentum  $m$  and angular velocity  $\Omega_H = a/(r_H^2 + a^2)$  respectively. The boson normalization factor is assumed to be included in the radial function  $R(r)$  and consequently in the boson amplitudes  $A_B^{(\text{inc})}$ ,  $A_B^{(\text{ref})}$  and  $A_B^{(\text{trans})}$ .

Next we consider the scattering problem for the spin 1/2 spinor field in Kerr geometry. We can write the spinor wave function in the form:

$$\psi(x) = \frac{1}{(\Delta \Sigma)^{1/4}} \begin{pmatrix} F(r) \mathcal{Y}(\theta, \phi) \\ iG(r) \mathcal{Y}'(\theta, \phi) \end{pmatrix}, \quad (4.19)$$

where  $\mathcal{Y}(\theta, \phi)$ ,  $\mathcal{Y}'(\theta, \phi)$  stand for normalized spin-angular functions. Asymptotic radial component solutions for infinity and near event horizon are obtained

$$F(r) \sim \sqrt{\frac{\omega + \mu}{\omega}} (A_F^{(\text{inc})} \exp(-ip_\infty r^*) + A_F^{(\text{ref})} \exp(ip_\infty r^*)), \quad (4.20)$$

$$G(r) \sim \frac{1}{(\omega + \mu)} \frac{d}{dr^*} F(r), \quad (4.21)$$

for infinity ( $r \rightarrow \infty$ ) and

$$F(r) \sim A_F^{(\text{trans})} \exp(-ip_H r^*), \quad (4.22)$$

$$G(r) \sim iF(r), \quad (4.23)$$

for near event horizon ( $r \rightarrow r_H$ ), where  $A_F^{(\text{inc})}$ ,  $A_F^{(\text{ref})}$  and  $A_F^{(\text{trans})}$  denote fermionic amplitudes of incident wave, reflected and transmitted waves respectively. Fermionic normalization factors are included in radial amplitudes as  $\sqrt{(\omega + \mu)/\omega}$  for infinity and 1 for near horizon because particles behave as massless ones.

We can derive the current conservation relation of the radial component between the infinity and event horizon

$$\frac{p_\infty}{\omega} (|A_B^{(\text{inc})}|^2 - |A_B^{(\text{ref})}|^2) = \frac{p_H}{|p_H|} |A_B^{(\text{trans})}|^2, \quad (4.24)$$

for scalar bosons and

$$\frac{p_\infty}{\omega} (|A_F^{(\text{inc})}|^2 - |A_F^{(\text{ref})}|^2) = |A_F^{(\text{trans})}|^2, \quad (4.25)$$

for spinors. The factor in front of  $|A_B^{(\text{trans})}|$  is different of the factor in front of  $|A_F^{(\text{trans})}|$ , which is consistent for the massless neutrino cases.

In order to relate the bosonic and fermionic current, we apply the Bargmann-Wigner formulation at radial infinity, which is considered as free flat spacetime. We obtain the relation of conserved current between scalar boson and spinor at infinity region ( $r \rightarrow \infty$ ) using equation (4.10) as

$$|A_B^{(\text{inc})}|^2 - |A_B^{(\text{ref})}|^2 = |A_F^{(\text{inc})}|^2 - |A_F^{(\text{ref})}|^2. \quad (4.26)$$

Combining three conserved current relations in equations (4.24) -(4.26), we get the conclusion that the momentum near the event horizon is positive:

$$0 < p_H = \omega - \Omega_H m. \quad (4.27)$$

This condition means that bosonic superradiance does not occur for  $0 < \omega$ .

It is worthwhile to note that bosonic and fermionic superradiance can occur for  $\omega < 0$  with  $0 < \omega - \Omega_H m$ .

These conclusion do not change by the choice of normalization factors because they are positive definite.

## 5 Summary

We have studied the super-radiance puzzle between bosons and fermions in rotating black hole geometry using the generalized Bargmann-Wigner formulation.

- We have obtained the direct wave function relation between scalar bosons and spinors via bi-spinor fields under the extended Bargmann-Wigner formulation, established in the paper.

- The superradiance phenomena for scalar bosons of the type  $0 < \omega$  and  $\omega - m\Omega_H < 0$  are shown not to occur in Kerr metric as those for fermion cases.

Some discussions are added. We have established the Bargmann-Wigner formulation for non-zero particle mass cases. We can take the massless limit after intermediate calculations because the mass is used as the regularization of the theory. We expect that the superradiance problem of vector and tensor bosons will be solved rigorously as in the scalar boson case.

## Acknowledgements

I would like to give my thanks to Professor Kazuyasu Shigemoto for his discussions and careful reading to this manuscript.

## Appendix: Gamma matrices in flat and curved spacetime

Gamma matrix notation used in this paper is given in flat Minkowski spacetime:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & -i\sigma_a \\ i\sigma_a & 0 \end{pmatrix}, \quad (a = 1 \sim 3)$$

where  $\sigma_a$  stand for Pauli matrices. Gamma matrices in curved spacetime are connected by the vierbeins as  $\gamma^\mu = b_i^\mu \gamma^i$ , where Latin suffix  $i = 0 \sim 3$  are for flat spacetime and Greek suffix  $\mu$  are for curved spacetime. From the vierbein condition  $\mathcal{D}_\mu b_\nu^i = 0$  in equation (3.12), differential equations for gamma matrices in curved spacetime are obtained:

$$\partial_\mu \gamma^\nu = -\Omega_\mu \gamma^\nu + \gamma^\nu \Omega_\mu - \Gamma_{\lambda\mu}^\nu \gamma^\lambda = -\omega_{ij\mu} \gamma^i b^{j\nu} - \Gamma_{\lambda\mu}^\nu \gamma^\lambda,$$

where the connection matrix  $\Omega_{\lambda\mu}^\nu$  and the spin connection  $\omega_{\mu}^{ij}$  are defined in equations (3.7) and (3.13) respectively.

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